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ASYMPTOTIC SOLUTION OF THE PROBLEM ON THE TEMPERATURE FIELD IN A WELL WITH ACCOUNT FOR THE RADIAL-VELOCITY DISTRIBUTION

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The problem on the temperature field of a liquid in a well has been asymptotically solved for the case of a constant temperature gradient in it. Expressions for calculating the temperature of a liquid in a well in the zero and first approximations have been obtained. It is shown that the asymptotic solution of the problem considered in the zero approximation makes it possible to obtain an algorithm involving averaging of the temperature of a liquid in a well in the case where the velocity profile of the liquid is constant. The space and time temperature distributions of a liquid in a well have been calculated, and the contribution of various physical processes to them has been analyzed.

The study of the nonstationary temperature fields in a liquid or a gas flowing in pipes is a fundamental problem of thermal physics because knowledge of these fields is necessary for calculating various engineering units, especially those that are used in pipeline transport and in oil and gas wells. At present, the possibility of calculating the temperature θ of a liquid or a gas in a well without regard for the radial distribution of their velocity in the well exists [1–5]; however, there is no theory that would allow one to calculate the temperature of a liquid in a well with account for the actual distribution of its velocity. A modified asymptotic method for solving the main problem of temperature well logging has been proposed and used in [3–5]. This method made it possible to determine the radial distribution of the temperature in the shaft of a working well.

The aim of the present work is to asymptotically solve the problem on the temperature field in the shaft of a working well with account for the velocity profile of a liquid or a gas in the well for the case of a constant temperature gradient in it.

As in previous works, it was assumed that the environment is homogeneous and anisotropic, the temperature θ_1 in distant rocks changes linearly with depth, and the seasonal variations in the temperature on the surface do not influence the deep regions considered. It was also assumed for simplicity that the temperature gradient remains unchanged: $\partial\theta_1/\partial z_d = \partial\theta/\partial z_d = -\Gamma$, the velocity of a liquid in the shaft of a well depends on the distance to the well axis $v = v_0 R(r_d)$ and the derivative of the temperature with respect to the radial coordinate on the z_d axis of the cylindrical coordinate system, pointing upwards at the center of the well, is equal to zero (symmetry condition).

Mathematical Formulation of the Problem. The mathematical model considered includes the equation for heat conductivity in the mass surrounding the shaft of the well

$$\rho_1 c_1 \frac{\partial\theta_1}{\partial\tau} = \lambda_{1r}\frac{1}{r_d}\frac{\partial}{\partial r_d}\left(r_d\frac{\partial\theta_1}{\partial r_d}\right), \quad r_d > r_0\,, \quad \tau > 0\,, \tag{1}$$

and the equation for convective heat exchange between the liquid (which is multiphase in the general case) and the heat sources in the shaft

$$\rho c\frac{\partial\theta}{\partial\tau} = \lambda_r\frac{1}{r_d}\frac{\partial}{\partial r_d}\left(r_d\frac{\partial\theta}{\partial r_d}\right) + \rho c v_0 R\,(r_d)\,\Gamma + q_1\,, \quad r_d < r_0\,, \quad \tau > 0\,. \tag{2}$$

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The density of sources is determined by the expression

$$q_1 = -\eta c\rho^2 g v_0 R\,(r_d) + q_d\,, \tag{3}$$

in which the term $-\eta c\rho^2 g v_0 R(r_d)$ defines the adiabatic effect in the rising stream and q_d accounts for the effects from the other heat sources, e.g., the temperature effect of the phase transitions caused by the liberation of gas. On condition that the temperature gradient remain unchanged, the second derivative of the temperature with respect to the coordinate z_d in Eqs. (1) and (2) is equal to zero. It was assumed that, at the interface between the shaft and the surrounding mass, the temperatures and heat flows are equal:

$$\theta\big|_{r_d=r_0} = \theta_1\big|_{r_d=r_0} \tag{4}$$

$$\lambda_r \frac{\partial\theta}{\partial r_d}\bigg|_{r_d=r_0} = \lambda_{1r}\frac{\partial\theta_1}{\partial r_d}\bigg|_{r_d=r_0}\,. \tag{5}$$

The initial conditions correspond to the nonperturbed, natural earth temperature that increases linearly with increase in the depth z_d

$$\theta\big|_{\tau=0} = \theta_1\big|_{\tau=0} = \theta_{01} - \Gamma z_d\,, \tag{6}$$

and is equal to the temperature at the points of the surrounding mass that are positioned at a large distance from the shaft

$$\theta_1\big|_{r_d\to\infty} = \theta_{01} - \Gamma z_d\,. \tag{7}$$

It was assumed that the velocity of the liquid in the shaft depends on the distance to the axis of the well:

$$v = v_0 R\,(r_d)\,. \tag{8}$$

Taking into account the concrete dependence of the velocity of the liquid on the radial coordinate, we constructed computational formulas for laminar and turbulent flows.

Using the relations $r = r_d/r_0$, $z = z_d/D$, $t = \tau\lambda_{1r}/(\rho_1 c_1 r_0^2)$, $\varepsilon = \lambda_{1r}/\lambda_r$, $T_1 = (\theta_1 - \theta_{01} + \mathrm{Gr} z_d)/\theta_0$, $T = (\theta - \theta_{01} + \mathrm{Gr} z_d)/\theta_0$, $\chi = c_1\rho_1/(c\rho)$, and $\gamma = r_0/D$, we brought problem (1)–(8) to the dimensionless form

$$\frac{\partial T_1}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T_1}{\partial r}\right), \tag{9}$$

$$\frac{\partial T}{\partial t} = \frac{\chi}{\varepsilon}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + (G - H)\,R\,(r) + Q\,(r, t)\,, \tag{10}$$

$$T\big|_{r=1} = T_1\big|_{r=1}\,, \tag{11}$$

$$\frac{\partial T}{\partial r}\bigg|_{r=1} = \varepsilon\frac{\partial T_1}{\partial r}\bigg|_{r=1}\,, \tag{12}$$

$$T\big|_{t=0} = T_1\big|_{t=0} = 0\,, \tag{13}$$

$$T_1\big|_{r\to\infty}=0\,, \tag{14}$$

where G = Pe$\gamma\Gamma D/\theta_0$, Pe = $v_0 r_0/a_{1r}$, $Q(r, t) = \chi r_0^2 q_d/(\theta_0\lambda_{1r})$, and H = Pe$\eta\rho g r_0/\theta_0$. From these formulas follows that it is appropriate to use ΓD for the quantity θ_0.

It is very difficult to analytically solve the problem in this formulation. Therefore the asymptotic method was used for its solution. We introduced the asymptotic-expansion parameter $\varepsilon = \lambda_{1r}/\lambda_r$, where λ_{1r} is the heat conductivity of the rock and λ_r is the turbulent heat conductivity of the liquid flow. Under actual conditions, the turbulent heat conductivity, which is due to the movement of liquid regions relative to each other, substantially exceeds (by 2–3 times) the heat conductivity in the rock [4]. In a turbulent flow, the asymptotic expansion parameter is small and equal to $\varepsilon \sim 0.01$–0.001 [4]. However, the parameter ε need not be small since the radius of convergence increases with time as $\sqrt{t}$. This provides an acceptable accuracy of calculations at $\varepsilon \sim 1$ and makes it possible to use the solutions, presented below, obtained in the zero and first approximations for a laminar flow. As the expansion parameter, the formal parameter ε', introduced artificially by the cofactor λ_{1r}/λ_r, can be used, which makes the substantiation of the smallness of the ratio between the heat conductivities unnecessary and makes it possible to obtain results coincident with the results presented below.

Problem (9)–(14) was solved by asymptotic expansion in terms of the parameter ε

$$T = T^{(0)} + \varepsilon T^{(1)} + \varepsilon^2 T^{(2)} + \dots\,, \tag{15}$$

$$T_1 = T_1^{(0)} + \varepsilon T_1^{(1)} + \varepsilon^2 T_1^{(2)} + \dots\,, \tag{16}$$

where the subscripts of the dimensionless temperature T denote the number of a region and the superscripts denote the ordinal number of an approximation. Substitution of (16) into (9) gives

$$\frac{\partial T_1^{(0)}}{\partial t} - \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T_1^{(0)}}{\partial r}\right) + \sum_{i=1}^{\infty} \varepsilon^i \left[\frac{\partial T_1^{(i)}}{\partial t} - \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T_1^{(i)}}{\partial r}\right)\right] = 0\,. \tag{17}$$

In a similar manner, from Eqs. (10) and (15) we find

$$-\chi\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T^{(0)}}{\partial r}\right) + \sum_{i=1}^{\infty} \varepsilon^i \left[\frac{\partial T^{(i-1)}}{\partial t} - \frac{\chi}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T^{(i)}}{\partial r}\right) - [R(r)(G-H) + Q(r,t)]\,\delta_{1,i}\right] = 0\,. \tag{18}$$

The boundary conditions take the form

$$\left(T^{(0)}\big|_{r=1} - T_1^{(0)}\big|_{r=1}\right) + \sum_{i=1}^{\infty} \varepsilon^i \left(T^{(i)}\big|_{r=1} - T_1^{(i)}\big|_{r=1}\right) = 0\,, \tag{19}$$

$$\left.\frac{\partial T^{(0)}}{\partial r}\right|_{r=1} + \sum_{i=1}^{\infty} \varepsilon^i \left(\left.\frac{\partial T^{(i)}}{\partial r}\right|_{r=1} - \left.\frac{\partial T_1^{(i-1)}}{\partial r}\right|_{r=1}\right) = 0\,, \tag{20}$$

$$T^{(0)}\big|_{t=0} + \sum_{i=1}^{\infty} \varepsilon^i\, T^{(i)}\big|_{t=0} = 0\,, \quad T_1^{(0)}\big|_{t=0} + \sum_{i=1}^{\infty} \varepsilon^i\, T_1^{(i)}\big|_{t=0} = 0\,, \tag{21}$$

$$T_1^{(0)}\big|_{r\to\infty} + \sum_{i=1}^{\infty} \varepsilon^i\, T_1^{(i)}\big|_{r\to\infty} = 0\,. \tag{22}$$

Formulation of the Problem in the Zero Approximation. It is assumed that ε = 0 in (18); then

$$-\frac{\chi}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T^{(0)}}{\partial r}\right)=0\,. \tag{23}$$

From (23) we obtain the expression for the radial temperature gradient in the zero approximation:

$$\frac{\partial T^{(0)}}{\partial r}=\frac{C_1}{r}\,. \tag{24}$$

It is assumed that the solution is limited at $r = 0$. In this case, it follows from (24) that the integration constant C_1 is equal to zero; therefore,

$$\frac{\partial T^{(0)}}{\partial r}=0\,. \tag{25}$$

It follows from (25) that, in the zero approximation, the temperature is independent of the radial coordinate r and is a function of the time t:

$$T^{(0)}=T^{(0)}\,(t)\,. \tag{26}$$

Assuming that ε = 0 in (17), we obtain the expression for the zero expansion coefficient $T_1^{(0)}$:

$$\frac{\partial T_1^{(0)}}{\partial t}-\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T_1^{(0)}}{\partial r}\right)=0\,. \tag{27}$$

Using (23), from (18) we obtain the equation for the zero coefficient of expansion of the temperature in the shaft:

$$\frac{\chi}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T^{(1)}}{\partial r}\right)=\frac{\partial T^{(0)}}{\partial t}-R\,(r)\,(G-H)-Q\,(r,t)\,. \tag{28}$$

Equation (28) is "coupled" because it contains the expansion coefficients of the zero and first orders $T^{(0)}$ and $T^{(1)}$. This makes the solution of the corresponding problems difficult. Below are transformations that make it possible to uncouple Eq. (28) by elimination of the term $T^{(1)}$ from it. Equation (28) will take the form

$$\frac{\chi}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T^{(1)}}{\partial r}\right)=A_1\,(t)-R\,(r)\,A_2\,(t)-Q\,(r,t)\,, \tag{29}$$

where the coefficients A_1 and A_2 represent time functions independent of the radial coordinate r:

$$A_1\,(t)=\frac{\partial T^{(0)}}{\partial t}\,,\quad A_2\,(t)=G-H\,. \tag{30}$$

Integrating (29), we obtain an expression for the radial derivative of the first coefficient of the asymptotic expansion

$$\frac{\partial T^{(1)}}{\partial r}=\frac{rA_1\,(t)}{2\chi}-\frac{A_2\,(t)\,R_1\,(r)}{r\chi}-\frac{Q_1\,(r,t)}{r\chi}\,. \tag{31}$$

On condition (20), from (31) follows that

$$\left.\frac{\partial T^{(1)}}{\partial r}\right|_{r=1} = \left.\frac{\partial T_1^{(0)}}{\partial r}\right|_{r=1} = \frac{A_1(t)}{2\chi} - \frac{A_2(t)\,R_1(1)}{\chi} - \frac{Q_1(1,t)}{\chi}\,, \tag{32}$$

where $R_1(r) = \int_0^r r'R(r')dr'$ and $Q_1(r,\ t) = \int_0^r r'Q(r',\ t)dr'$. Substitution of (30) into (32) gives an equation containing only the zero-order expansion coefficients:

$$\frac{\partial T^{(0)}}{\partial t} - 2\,(G-H)\,R_1(1) - 2Q_1(1,t) = 2\chi \left.\frac{\partial T_1^{(0)}}{\partial r}\right|_{r=1}, \quad r<1\,, \quad t>0\,. \tag{33}$$

Thus, the above transformations allowed us to "uncouple" initial equation (28). The finite formulation of the problem in the zero approximation includes, along with (33), the following equations representing the initial and boundary conditions:

$$\frac{\partial T_1^{(0)}}{\partial t} - \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T_1^{(0)}}{\partial r}\right) = 0\,, \quad r>1\,, \quad t>0\,, \tag{34}$$

$$T^{(0)} = T_1^{(0)}\big|_{r=1}\,, \tag{35}$$

$$T^{(0)}\big|_{t=0} = T_1^{(0)}\big|_{t=0} = 0\,, \tag{36}$$

$$T_1^{(0)}\big|_{r\to\infty} = 0\,. \tag{37}$$

Problem (33)–(37) is characterized by the existence of the trace of the derivative of the temperature with respect to the outer region in Eq. (33).

Boundary-Value Problem for the First Expansion Coefficients. For the first expansion coefficient, Eq. (18) takes the form

$$\frac{\partial T^{(1)}}{\partial t} - \frac{\chi}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T^{(2)}}{\partial r}\right) = 0\,, \quad r<1\,, \quad t>0\,, \tag{38}$$

and is "coupled" because includes the first- and second-order expansion coefficients. Let us "uncouple" Eqs. (38). From (31) we obtain

$$T^{(1)} = \frac{r^2}{4\chi}A_1(t) - \frac{1}{\chi}A_2(t)\,R_2(r) - \frac{1}{\chi}\,Q_2(r,t) + B(t)\,. \tag{39}$$

Substitution of (39) into (38) gives the expression for the radial derivative of the second expansion coefficient:

$$\frac{\partial T^{(2)}}{\partial r} = \frac{r^3}{16\chi^2}\frac{\partial A_1(t)}{\partial t} - \frac{R_3(r)}{\chi^2}\frac{\partial A_2(t)}{\partial t} - \frac{1}{\chi^2}\frac{\partial Q_3(r,t)}{\partial t} + \frac{r}{2\chi}\frac{\partial B(t)}{\partial r}\,, \quad r<1\,, \quad t>0\,, \tag{40}$$

where

$$R_2(r) = \int_0^r r'^{-1}\,R_1(r')\,dr'\,; \quad R_3(r) = r^{-1}\int_0^r r'\,R_2(r')\,dr'\,; \quad Q_2(r,t) = \int_0^r r'^{-1}\,Q_1(r',t)\,dr'\,;$$

$$Q_3(r,t) = r^{-1}\int\limits_0^r r' Q_2(r',t)\,dr' .$$

Using condition (20) at $i = 1$, we obtain

$$\left.\frac{\partial T^{(2)}}{\partial r}\right|_{r=1} = \frac{1}{16\chi^2}\frac{\partial A_1(t)}{\partial t} - \frac{R_3(1)}{\chi^2}\frac{\partial A_2(t)}{\partial t} - \frac{1}{\chi^2}\frac{\partial Q_3(1,t)}{\partial t} + \frac{1}{2\chi}\frac{\partial B(t)}{\partial t} = \left.\frac{\partial T_1^{(1)}}{\partial r}\right|_{r=1} . \tag{41}$$

If the initial conditions are known, one can determine the one unknown coefficient $B(t)$ in Eq. (41) and, by doing so, determine $T^{(1)}$ and thus solve the problem considered. However, it makes sense to formulate the boundary-value problem for the first expansion coefficient $T^{(1)}$. Differentiating (39) over t and substituting the expression obtained into (41), we obtain, using (30), the finite equation for the first expansion coefficient, containing the imputation of the derivative of the first expansion coefficient with respect to the environment:

$$\frac{\partial T^{(1)}}{\partial t} + \frac{1-2r^2}{8\chi}\frac{\partial^2 T^{(0)}}{\partial t^2} + \frac{1}{\chi}\frac{\partial\,[Q_2(r,t) - 2Q_3(1,t)]}{\partial t} = 2\chi\left.\frac{\partial T_1^{(1)}}{\partial r}\right|_{r=1} , \quad r<1 , \quad t>0 . \tag{42}$$

The mathematical model of the problem for the first expansion coefficients also includes the equation for the environment

$$\frac{\partial T_1^{(1)}}{\partial t} - \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T_1^{(1)}}{\partial r}\right) = 0 , \quad r>1 , \quad t>0 , \tag{43}$$

the condition at the interface

$$T^{(1)}\big|_{r=1} = T_1^{(1)}\big|_{r=1} \tag{44}$$

and the condition at infinity

$$T_1^{(1)}\big|_{r\to\infty} = 0 . \tag{45}$$

The problem considered is solved in the form of (39). In this case, the initial conditions that were formulated for medium temperatures or for the temperatures at given points must be changed:

$$\langle T^{(1)}\rangle\big|_{t=0} = T_1^{(1)}\big|_{t=0} = 0 , \quad T^{(1)}\big|_{r=r_1,t=0} = T_1^{(1)}\big|_{t=0} = 0 . \tag{46}$$

The initial condition or the value of the coordinate r_1 at which this condition is fulfilled is determined additionally.

Solution of the Problem in the Zero Approximation. Using the Laplace–Carson transform

$$T_j^{(0)\mathrm{im}} = p\int\limits_0^\infty \exp(-pt)\,T_j^{(0)}(t)\,dt \tag{47}$$

we write problem (33)–(37) in the image space

$$pT_1^{(0)\mathrm{im}} - \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T_1^{(0)\mathrm{im}}}{\partial r}\right) = 0 , \quad r>1 , \quad t>0 , \tag{48}$$

$$pT^{(0)\text{im}} - 2\,(G-H)\,R_1\,(1) - 2Q_1^{\text{im}}\,(1, p) = 2\chi \left.\frac{\partial T_1^{(0)\text{im}}}{\partial r}\right|_{r=1}, \quad r < 1\,, \quad t > 0\,, \tag{49}$$

$$T^{(0)\text{im}} = T_1^{(0)\text{im}}\big|_{r=1}\,, \tag{50}$$

$$T_1^{(0)\text{im}}\big|_{r\to\infty} = 0\,. \tag{51}$$

The solution of Eq. (48) is expressed in terms of the zero-order Bessel function of the imaginary argument. Using (50), we obtain

$$T_1^{(0)\text{im}} = \frac{K_0\,(r\sqrt{p})}{K_0\,(\sqrt{p})}\,T^{(0)\text{im}}\,, \tag{52}$$

$$\left.\frac{\partial T_1^{(0)\text{im}}}{\partial r}\right|_{r=1} = -\sqrt{p}\,\frac{K_1\,(\sqrt{p})}{K_0\,(\sqrt{p})}\,T^{(0)\text{im}} = -\sqrt{p}\,kT^{(0)\text{im}}\,, \tag{53}$$

where $k = k(p) = K_1(\sqrt{p})/K_0(\sqrt{p})$. Taking into account (53), we represent Eq. (49) for $T^{(0)\text{im}}$ in the form

$$pT^{(0)\text{im}} - 2\,(G-H)\,R_1\,(1) - 2Q_1^{\text{im}}\,(1, p) = -2\chi k\sqrt{p}\,T^{(0)\text{im}}\,, \quad r < 1\,. \tag{54}$$

Equation (54) is solved as

$$T^{(0)\text{im}} = 2\,\frac{(G-H)\,R_1\,(1) + Q_1^{\text{im}}\,(1, p)}{p + 2\chi k\sqrt{p}}\,, \quad r < 1\,. \tag{55}$$

Substitution of (55) into (52) gives the solution for the outer region

$$T_1^{(0)\text{im}} = 2\,\frac{K_0\,(r\sqrt{p})}{K_0\,(\sqrt{p})}\left[\frac{(G-H)\,R_1\,(1) + Q_1^{\text{im}}\,(1, p)}{p + 2\chi k\sqrt{p}}\right], \quad r > 1\,. \tag{56}$$

Expressions (55) and (56) represent an exact solution of the problem in the zero approximation in the image space. It makes it possible to determine the temperatures averaged over the cross section of a well. This solution is identical, in the particular case of a constant velocity profile in the absence of sources, to the solution presented in [2, 3]. It follows herefrom that, due to the appropriate choice of the asymptotic expansion parameter, the solution in the zero approximation can be used for calculating the temperature for large and small times, despite the fact that the method considered provides obtaining results only for large times.

Construction of the Solution for the First Expansion Coefficients. In the Laplace–Carson image space, problem (42)–(46) for the first expansion coefficients takes the form

$$pT_1^{(1)\text{im}} - \frac{1}{r}\frac{\partial}{\partial r}\left(r\,\frac{\partial T_1^{(1)\text{im}}}{\partial r}\right) = 0\,, \quad r > 1\,, \tag{57}$$

$$p\left(T^{(1)\text{im}} - T^{(1)}\,(0)\right) + \frac{1-2r^2}{8\chi}\left[p^2 T^{(0)\text{im}} - p\,\frac{\partial T^{(0)}}{\partial t}\,(t=0)\right] +$$

$$+\frac{p}{\chi}\left[Q_2^{\text{im}}(r,p)-2Q_3^{\text{im}}(1,p)-Q_2(r,0)+2Q_3(1,0)\right]=2\chi\left.\frac{\partial T_1^{(1)\text{im}}}{\partial r}\right|_{r=1},\quad r<1\,, \tag{58}$$

$$T^{(1)\text{im}}\big|_{r=1}=T_1^{(1)\text{im}}\big|_{r=1}\,, \tag{59}$$

$$T_1^{(1)\text{im}}\big|_{r\to\infty}=0\,. \tag{60}$$

The solution of Eq. (57) is expressed in terms of the zero-order Bessel function. By analogy with the zero approximation, we represent the solution for an oil pool and its derivative at the boundary $r = 1$:

$$T_1^{(1)\text{im}}=T^{(1)\text{im}}\big|_{r=1}\frac{K_0(r\sqrt{p})}{K_0(\sqrt{p})}\,,\qquad \left.\frac{\partial T_1^{(1)\text{im}}}{\partial r}\right|_{r=1}=-\sqrt{p}\,kT^{(1)\text{im}}\big|_{r=1}\,. \tag{61}$$

The rate of change in the temperature at the initial instant of time $\partial T^{(0)}/\partial t$ ($t = 0$) is determined from Eq. (33). Substituting (61) into (58), we obtain the equation for $T^{(1)\text{im}}$:

$$p\left(T^{(1)\text{im}}-T^{(1)}(0)\right)+\frac{1-2r^2}{8\chi}\left[p^2T^{(0)\text{im}}-2p\left((G-H)R_1(1)+Q_1(1,0)\right)\right]+$$

$$+\frac{p}{\chi}\left[Q_2^{\text{im}}(r,p)-2Q_3^{\text{im}}(1,p)-Q_2(r,0)+2Q_3(1,0)\right]=-2\chi k\sqrt{p}\,T^{(1)\text{im}}\big|_{r=1}\,. \tag{62}$$

From (42) and (31), it follows that

$$T^{(1)\text{im}}=\frac{r^2}{4\chi}pT^{(0)\text{im}}-\frac{R_2(r)}{\chi}(G-H)-\frac{1}{\chi}Q_2^{\text{im}}(r,p)+B^{\text{im}}(p)\,. \tag{63}$$

Substituting (63) into (62), we determine the coefficient $B^{\text{im}}(p)$ in the image space:

$$B^{\text{im}}(p)=(\sqrt{p}+2\chi k)^{-1}\Big\{\sqrt{p}\,B(0)-\frac{p}{8\chi}(\sqrt{p}+4\chi k)\,T^{(0)\text{im}}+\frac{1}{4\chi}(\sqrt{p}\,R_1(1)+8\chi kR_2(1))\,(G-H)+$$

$$+\frac{\sqrt{p}}{4\chi}Q_1(1,0)+2kQ_2^{\text{im}}(1,p)+\frac{2\sqrt{p}}{\chi}[Q_3^{\text{im}}(1,p)-Q_3(1,0)]\Big\}\,. \tag{64}$$

Substitution of (64) into (63) gives the expression for the first expansion coefficient in the well:

$$T^{(1)\text{im}}=\frac{1}{8\chi}\left(2r^2-1-\frac{2\chi k}{\sqrt{p}+2\chi k}\right)pT^{(0)\text{im}}+\left[1+\frac{2\chi k}{\sqrt{p}+2\chi k}\left(4\cdot\frac{R_2(1)}{R_1(1)}-1\right)-4\cdot\frac{R_2(r)}{R_1(1)}\right]\frac{G-H}{4\chi}R_1(1)+$$

$$+\frac{1}{\chi}Q_2^{\text{im}}(r,p)+(\sqrt{p}+2\chi k)^{-1}\left[\sqrt{p}\,B(0)+\frac{\sqrt{p}}{4\chi}Q_1(1,0)+2kQ_2^{\text{im}}(1,p)+\frac{2\sqrt{p}}{\chi}\left[Q_3^{\text{im}}(1,p)-Q_3(1,0)\right]\right]. \tag{65}$$

Using (65) and (61), we determine the expression for the temperature in the environment:

$$T_1^{(1)\text{im}}=\frac{K_0(r\sqrt{p})}{K_0(\sqrt{p})}\left\{\frac{1}{8\chi}\frac{\sqrt{p}}{\sqrt{p}+2\chi k}\left[pT^{(0)\text{im}}+2R_1(1)\left(1-4\cdot\frac{R_2(1)}{R_1(1)}\right)(G-H)\right]+\frac{1}{\chi}Q_2^{\text{im}}(r,p)+(\sqrt{p}+2\chi k)^{-1}\times\right.$$

$$\times\left[\sqrt{p}\,B(0)+\frac{\sqrt{p}}{4\chi}Q_1(1,0)+2kQ_2^{\mathrm{im}}(1,p)+\frac{2\sqrt{p}}{\chi}\left[Q_3^{\mathrm{im}}(1,p)-Q_3(1,0)\right]\right]\Bigg\}. \tag{66}$$

Expressions (65) and (66) represent exact solutions of the problem for the first expansion coefficient, where $T^{(0)\mathrm{im}}$ is determined from expression (55). The value of $B(0)$ is dependent on initial conditions (46) and is determined by the formulas

$$B(0)=\frac{8R_3(1)-1}{4\chi}(G-H)-\frac{1}{4\chi}Q_1(1,0)+\frac{2}{\chi}Q_3(1,0),$$
$$B(0)=\frac{2R_2(r_1)-r_1^2}{2\chi}(G-H)-\frac{r_1^2}{2\chi}Q_1(1,0)+\frac{1}{\chi}Q_2(r_1,0). \tag{67}$$

However, according to (39), the choice of the initial conditions does not influence the radial distribution of the temperature in the well. Expression (65) defines a developed temperature profile; therefore, the range of applicability of the first approximation is limited to fairly large times. This restriction does not hold for the zero approximation because it determines the temperatures averaged over the cross section of the well.

If v_0 is the average velocity, the integral $R_1(1) = 1/2$. In this case, problem (33)–(37) and its solution for the zero approximation (55), (56) are identical to the corresponding problem and its solution for a constant velocity independent of the radial coordinate. This is a mathematical substantiation of the statement, used earlier, that, in such problems, the velocity averaged over the cross section of a well is constant independently of the radial profile. Hence it follows that the obtaining of the limiting solution as $\varepsilon \to 0$ is equivalent to the averaging of the desired solution. Note that, in the case considered, it is difficult to average the initial problem by direct integration because of the dependence of the velocity on the radial coordinate. The use of the asymptotic method in the case where the parameter p is selected in the above-described way provides a realization of the procedure of averaging of the desired solution over the cross section of the well. It should also be noted that it is impossible to compare the temperature profiles for the first expansion coefficient in an analogous way.

In the case where there are no sources in a well, the solutions of the corresponding problems are simpler and have the following form:

$$T^{(0)\mathrm{im}}=\frac{2R_1(1)(G-H)}{p+2\chi k\sqrt{p}},\quad r<1, \tag{68}$$

$$T_1^{(0)\mathrm{im}}=\frac{2R_1(1)(G-H)}{p+2\chi k\sqrt{p}}\frac{K_0(r\sqrt{p})}{K_0(\sqrt{p})},\quad r>1, \tag{69}$$

$$T^{(1)\mathrm{im}}=\frac{1}{8\chi}\left(2r^2-1-\frac{2\chi k}{\sqrt{p}+2\chi k}\right)pT^{(0)\mathrm{im}}+\left[1+\frac{2\chi k}{\sqrt{p}+2\chi k}\left(4\cdot\frac{R_2(1)}{R_1(1)}-1\right)-4\cdot\frac{R_2(r)}{R_1(1)}\right]\times$$
$$\times\frac{G-H}{4\chi}R_1(1)+\frac{\sqrt{p}\,B(0)}{\sqrt{p}+2\chi k}, \tag{70}$$

$$T_1^{(1)\mathrm{im}}=\frac{K_0(r\sqrt{p})}{K_0(\sqrt{p})}\left\{\frac{1}{8\chi}\frac{\sqrt{p}}{\sqrt{p}+2\chi k}\left[pT^{(0)\mathrm{im}}+2R_1(1)(G-H)\left(1-4\cdot\frac{R_2(1)}{R_1(1)}\right)\right]+\frac{\sqrt{p}\,B(0)}{\sqrt{p}+2\chi k}\right\}. \tag{71}$$

The solutions obtained allow one to calculate the dependences of the temperature in a well on different parameters. By way of example, we present formulas for the temperature in a well in the zero and first approximations for fairly small times $k \approx 1$:

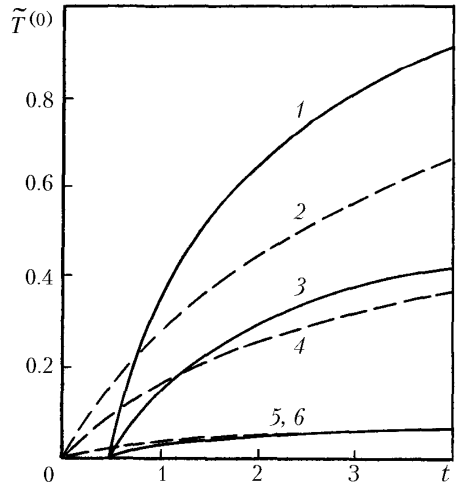

Fig. 1. Dependence of the relative temperature $\tilde{T}^{(0)} = T^{(0)}/(G-H)$ on the dimensionless time: χ = 0.6 (1, 2), 1.3 (3, 4), and 10 (5, 6) [1, 3, 5) in the large-time approximation; 2, 4, 6) in the small-time approximation].

$$T^{(0)} = \frac{2R_1(1)(G-H)}{\chi}\left\{\sqrt{\frac{t}{\pi}} - \frac{1}{4\chi}\left[1-\exp(4\chi^2 t)\,\mathrm{erfc}(2\chi\sqrt{t})\right]\right\}, \quad r<1\,, \tag{72}$$

$$T^{(1)} = \frac{G-H}{4\chi}R_1(1)\left\{\left(2r^2 - 4\chi t + \frac{1}{2\chi}\right)\exp(4\chi^2 t)\,\mathrm{erfc}(2\chi\sqrt{t}) - 2\sqrt{\frac{t}{\pi}} + 1 - 4\cdot\frac{R_2(r)}{R_1(1)} + \right.$$

$$\left. + \left(4\cdot\frac{R_2(1)}{R_1(1)} - 1\right)\left[1-\exp(4\chi^2 t)\,\mathrm{erfc}(2\chi\sqrt{t})\right] + B(0)\exp(4\chi^2 t)\,\mathrm{erfc}(2\chi\sqrt{t})\right\}. \tag{73}$$

According to (68)–(71), the zero approximation determines the average values of the temperature, and the radial distribution of the temperature in the well is determined by the first approximation.

For small values of the parameter p or large times the relation between the Bessel functions $k \approx -\{\sqrt{p}[C + \ln(\sqrt{p}/2)]\}^{-1}$ is fulfilled. Using this expression, one can obtain simple asymptotic formulas for large times (C = 0.577 is the Euler constant):

$$T^{(0)} = \frac{R_1(1)(G-H)}{2\chi}(\ln 4t - C)\,, \quad r<1\,, \tag{74}$$

$$T^{(1)} = \frac{G-H}{4\chi}R_1(1)\left[(r^2-1)\frac{1}{2\chi t} + 4\cdot\frac{R_2(1)-R_2(r)}{R_1(1)} - \frac{2R_1(r_1)-r_1^2}{t}\right]. \tag{75}$$

From expression (65) follows the stationary temperature profile in the well

$$T^{(1)} = \frac{G-H}{\chi}\left[R_2(1) - R_2(r)\right]. \tag{76}$$

The expressions obtained by the asymptotic method allow one to calculate the temperature fields in a well. Figure 1 shows the calculated dependences of the relative temperature $\tilde{T}^{(0)} = T^{(0)}/(G-H)$ on the dimensionless time in the zero approximation for laminar flows of water with χ = 0.6, oil with χ = 1.3, and methane with χ = 10 in a well with $r < 1$. The dependences obtained in the approximation of small times (72) and large times (74) are shown

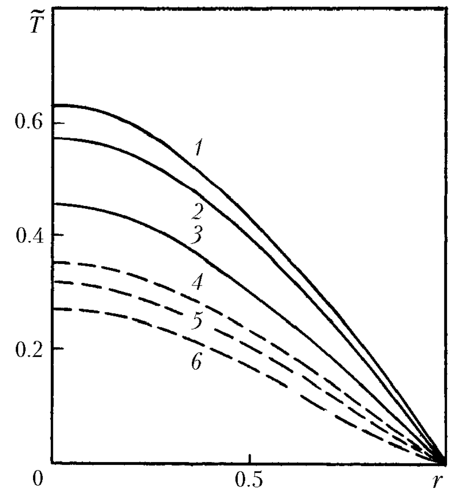

Fig. 2. Radial profiles of the relative temperature $\tilde{T} = [T^1(r) - T^1(r = 1)]/(G - H)$ in a laminar water flow in the well for large times [t = 100 (1), 3 (2), and 1 (3)] and small times [t = 0.1 (4), 0.05 (5), and 0.01 (6)].

respectively by the dashed and solid lines. Analysis of the curves obtained shows that the small-time approximation provides satisfactory results for the dimensionless times $t < 1$ and the large-time approximation should be used for practical calculations at $t > 1$. Note that the temperature of the gas or oil flowing in a well can be calculated with a high accuracy at $t < 10$ with the use of the small-time approximation.

Figure 2 shows the radial temperature profile of a laminar water flow in a well. In this case, the temperature profiles for small times ($t < 1$, dashed curves) were calculated using the first small-time approximation (73) and the temperature profiles for $t > 1$ were calculated using the first approximation for large times (75). It is seen from this figure that the drawback of the first approximation is that the temperature determined with it is dependent on the radial coordinate at small times. This points to the fact that the first approximation should be used for fairly large times, which is bound to be in accordance with the method used for solving the problem.

Thus, it has been shown that the distribution of the temperature fields over the cross section of a well can be estimated using asymptotic methods, which is of great practical importance.

NOTATION

A_1, A_2, B, G, H, Q_1, Q_2, Q_3, R_1, R_2, R_3, auxiliary functions and constants; a_{1r}, radial heat-conductivity coefficient of the rock, m^2/sec; C, Euler constant; C_1, integration constant; c, c_1, specific heat of the liquid and the rock, respectively, J/(K·kg); D, depth of the well, m; g, free-fall acceleration, m^2/sec; $K_0(x)$, $K_1(x)$, modified Bessel functions; Pe, Peclet parameter; p, parameter of the Laplace–Carson transform; $Q(r, t)$, dimensionless function of the density of heat sources; q_d, q_1, densities of heat sources, J/(sec·m^3); $R(r)$, radial velocity-distribution function; r_0, radius of the well, m; r_d, z_d, and r, z, dimensional and dimensionless cylindrical coordinates; T, T_1, dimensionless temperatures of the liquid flow and the rock; t, dimensionless time; $v(r)$, z coordinate of the velocity-field vector of the liquid, m/sec; v_0, normalization factor of the velocity-field vector of the liquid or average velocity of the liquid, m/sec; Γ, geothermal gradient, K/m; δ_{ij}, Kronecker delta; ε, asymptotic expansion parameter; λ_r, λ_{r1}, radial heat-conductivity coefficients of the liquid and the rock, W/(K·m); θ, θ_1, temperatures of the liquid and the surrounding rocks, K; θ_0, normalization temperature factor, K; θ_{01}, natural temperature of the rock at the origin of coordinates, K; ρ, ρ_1, densities of the liquid and the surrounding rocks, kg/m^3; τ, times, sec; η, adiabatic coefficient, K/Pa; γ, ratio between the radius of the well and its depth; χ, relative heat capacity per unit volume erfc $(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty \exp(-u^2)du$. Subscripts: 0, definite value of a variable; 1, surrounding rock; d, dimensional; *i*, ordinal number; *r*, *z*, directions; w, water; im, image; the superscript in parentheses corresponds to the ordinal number of the expansion; prime, integration variable.

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